

A way approaches to the infinitude of primes

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Abstract

To learn the Twin Primes Conjecture, the Goldbach Conjecture, etc. we introduce a way using elementary set theory to approach to the infinitude of primes.

The Greek mathematician Euclid (c.300,B.C.) firstly proved that the number of primes is infinite. The advantage of his proof was pretty short and depended immediately on the definition of the primes. During the over two thousand years after Euclid, some mathematician created different methods to the infinitude of primes. A popular one (P.59 of [2]) of the proofs was by the Swiss mathematician Leonhard Euler (1707 - 1783). His proof was by analytic method of infinite series and infinite product. In 1919, the Norwegian mathematician Viggo Brun (1885 - 1978) in his article [1] proved

$$\sum_{\substack{p \\ p+2 \text{ prime}}} \frac{1}{p} < \infty$$

where p is a prime. He was the first one to exhibit analytic method for solving such kind of questions of the Twin Primes Conjecture, the Goldbach Conjecture, etc. The question is where was Brun's great idea from? Was it from Euler?

The Twin Primes conjecture, Goldbach Conjecture, etc. of the central problems of mathematics are still unsolved now. We think that Brun's creative work remind, motivate and clue us that reviewing the all methods which were for approaching the infinitude of primes and creating some new ways are necessary and inevitable. In [3] we gave a way to approach the infinitude of primes as an application of Pritchard's sieve; in this article we offer another way base on the elementary set theory.

Let N be the natural number set. Saying $a, b \in N$ relatively prime if $(a, b) = 1$, where (a, b) represents the greatest common factor of a and b . Let $P = \{p_1, p_2, \dots\}$ represent the

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set of all prime numbers, where $p_1 = 2$, $p_2 = 3$ and $p_i < p_j$, if $i < j$. Denote $\pi_k = \prod_{i=1}^k p_i$, and $\pi_0 = 1$. Our goal is proving P is an infinite set.

Suppose P is finite, and its maximal element is p_n . Define

$$A = \{a \in N : 1 \leq a \leq \pi_n\}$$

and

$$A_k = \{a = bp_k, b \in N : (a, \pi_{k-1}) = 1, a \in A\}, \quad 1 \leq k \leq n,$$

then A_k , $1 \leq k \leq n$, are disjoint each other and subsets of A .

Let $|B|$ represents the cardinality of the set B . Then

$$\begin{aligned} |A_1| &= \frac{\pi_n}{p_1} = \pi_n \cdot \frac{1}{p_1}, \\ |A_2| &= \frac{\pi_n}{p_2} - \frac{\pi_n}{p_1 p_2} = \pi_n \cdot \left(\frac{1}{p_2} - \frac{1}{p_1 p_2} \right), \\ |A_3| &= \frac{\pi_n}{p_3} - \frac{\pi_n}{p_1 p_3} - \frac{\pi_n}{p_2 p_3} + \frac{\pi_n}{p_1 p_2 p_3} = \pi_n \cdot \left(\frac{1}{p_3} - \frac{1}{p_1 p_3} - \frac{1}{p_2 p_3} + \frac{1}{p_1 p_2 p_3} \right), \\ |A_4| &= \pi_n \cdot \left(\frac{1}{p_4} - \frac{1}{p_1 p_4} - \frac{1}{p_2 p_4} - \frac{1}{p_3 p_4} + \frac{1}{p_1 p_2 p_4} + \frac{1}{p_1 p_3 p_4} + \frac{1}{p_2 p_3 p_4} - \frac{1}{p_1 p_2 p_3 p_4} \right), \\ &\quad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ |A_n| &= \pi_n \cdot \left(\frac{1}{p_n} - \sum_{1 \leq i \leq n-1} \frac{1}{p_i p_n} + \sum_{1 \leq i < j < n} \frac{1}{p_i p_j p_n} - \dots + (-1)^{n-1} \frac{1}{p_1 \cdots p_{n-1} p_n} \right). \end{aligned}$$

Let $R = A - \bigcup_{i=1}^n A_i$. Notice

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{i=1}^n |A_i|,$$

since A_k , $1 \leq k \leq n$, are disjoint each other. Thus,

$$\begin{aligned} |R| &= \left| A - \bigcup_{i=1}^n A_i \right| \\ &= |A| - \left| \bigcup_{i=1}^n A_i \right| = \pi_n - \sum_{i=1}^n |A_i| \\ &= \pi_n \cdot \left(1 - \sum_{i=1}^n \frac{1}{p_i} + \sum_{1 \leq i < j \leq n} \frac{1}{p_i p_j} - \sum_{1 \leq i < j < k \leq n} \frac{1}{p_i p_j p_k} + \dots + (-1)^n \frac{1}{p_1 \cdots p_n} \right) \\ &= \pi_n \cdot \prod_{i=1}^n \left(1 - \frac{1}{p_i} \right) \qquad \text{(By the general form of the Binomial Theorem)} \end{aligned}$$

That $\pi_n \cdot \prod_{i=1}^n \left(1 - \frac{1}{p_i}\right) > 0$ implies $R \neq \phi$. By The Minimal Principle (P.3, [4]) that a nonempty subset of natural number set N contains its minimal element implies that $x = \min R \in R$.

We know that $x \notin A_i$, for all i , $1 \leq i \leq n$, which imply $x \neq p_i$, $1 \leq i \leq n$. Hence $x \notin P$. Thus, x is a composite number. Thus, there is an s , $1 \leq s \leq n$, such that p_s is the least prime factor of x . Then $x \in A_s$. Contradiction.

Hence the prime number set P is infinite.

References

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